

The many levels pairing Hamiltonian for two pairs

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Abstract. We address the problem of two pairs of fermions living on an arbitrary number of single-particle levels of a potential well (mean field) and interacting through a pairing force in the framework of the Richardson equations. The associated solutions are classified in terms of a number v_l , which reduces to the seniority v in the limit of a large pairing strength G and yields the number of pairs not developing a collective behaviour, their energy remaining finite in the $G \rightarrow \infty$ limit. We express analytically, through the moments of the single-particle levels distribution, the collective mode energy and the two critical values G_{cr}^+ and G_{cr}^- of the coupling which can exist on a single-particle level with no pair degeneracy. Notably G_{cr}^+ and G_{cr}^- , when the number of single particle levels goes to infinity, merge into the critical coupling of a one-pair system G_{cr} (when it exists), which is not envisioned by the Richardson theory. In correspondence of G_{cr} , the system undergoes a transition from a mean-field- to a pairing-dominated regime. We finally explore the behaviour of the excitation energies, wave functions and pair transfer amplitudes *versus* G finding out that the former, for $G > G_{\text{cr}}^-$, come close to the BCS predictions, whereas the latter display a divergence at G_{cr} , signaling the onset of a long-range off-diagonal order in the system.

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1 Introduction

In a previous paper [1] we examined in detail the problem of the fermionic pairing Hamiltonian \hat{H}_{pair} in the simple situation of one pair of nucleons coupled to an angular momentum $J = 0$ living in a set of L single-particle levels (s.p.l.), *e.g.* in a major shell of the nuclear mean field. This problem was solved long ago for the case of one s.p.l. hosting n pairs of fermions: yet it still presents aspects deserving further investigation in the non-degenerate case.

Actually, this is the situation one faces in applying \hat{H}_{pair} to real systems like nuclei and metals, which in fact represent two extreme situations of the non-degenerate case: in the former a major shell is typically split into five or six s.p.l. of different angular momenta; in the latter the number of non-degenerate levels entering into a band corresponds to a significant fraction of the Avogadro number. Moreover, in a heavy nucleus the number of pairs living in a level may be as large as, say, eight while in a metal it is one.

Recently, a renewed and widespread interest for \hat{H}_{pair} in the non-degenerate frame has flourished in connection with the issues of the physics of ultrasmall metallic grains, possibly superconducting [2], and of Bose-Einstein condensation [3].

When only one pair in L s.p.l. is considered, then the eigenstates of the pairing Hamiltonian fall into two classes: one collective and $L - 1$ trapped (in between the s.p.l.) states. In this connection it was found in [1] that:

1. the energy E_{coll} of the collective mode is related to the statistical features of the s.p.l. distribution, in the sense that only few moments of the latter, beyond of course the strength of the pairing interaction, are sufficient to accurately predict E_{coll} ;
2. the eigenvalues E_{tr}^ν ($\nu = 2, \dots, L$) of the states trapped in between the s.p.l. and hence constrained in the range $e_{\nu-1} < E_{\text{tr}}^\nu < e_\nu$, being the e_i ($i = 1, \dots, L$) the single-particle energies, belong to two different regimes: in the weak-coupling regime, for ν both small and large, the E_{tr}^ν almost coincide with e_ν (the more so, the smaller the degeneracy of the unperturbed s.p.l. is), whereas in the strong-coupling regime, *provided the degeneracy of the s.p.l. increases with ν* , the E_{tr}^ν , for

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small ν , almost coincide with $e_{\nu-1}$. Notably the transition between the two situations occurs more and more sharply, namely in correspondence of a precise value of the coupling constant G , as L increases.

In the present paper we extend the previous analysis considering two pairs of fermions living in many levels. First, we shall see that in this case the eigenstates of the pairing Hamiltonian are conveniently classified in terms of a number v_l , which provides a measure of the degree of collectivity of the states. This number is directly linked to the number N_G , first introduced by Gaudin in ref. [4], through the relation $v_l = 2N_G$ and might be considered as a sort of “like-seniority” (hence the notation v_l)¹, since it reduces to the standard seniority v for large G : as pointed out in [5], it has the significance of the number of pair energies (see below) which remain finite as G goes to infinity.

Futhermore for the $n = 2$ system we address the issues of:

1. expressing the energy E_{coll} of the collective $v_l = 0$ state in terms of the statistical features of the s.p.l. distribution, *i.e.* in terms of the moments of the latter;
2. exploring the existence and providing an analytic expression of the critical values of G (we already know from [6] that at most two of them might exist on a s.p.l.) which signal the transition of the system between two different regimes;
3. relating these values of G_{cr} to the one previously found in the $n = 1$ case;
4. studying the transition from the weak-coupling (where the mean field dominates) to the strong-coupling regime (where the pairing interaction dominates) by following the behaviour with G of the pair transfer matrix elements between one and two pairs states;
5. relating the exact and BCS solution in the quite extreme situation of few pairs (in fact two) and few s.p.l. (in fact three).

Anticipating our results, we will find that indeed, under suitable conditions, two values of G_{cr} exist. They turn out to coincide in the large- L limit and, notably, they coincide as well with the one found in ref. [1] for the $n = 1$ case. Their analytic expression can also be related, as for E_{coll} , to the moments of the distribution of the s.p.l. (in fact to the inverse of the latter) and they again mark the transition between the two above-mentioned regimes.

Indeed for $G > G_{\text{cr}}$, the exact solutions of the Richardson equations appear to come very close to the Bogoliubov quasi-particle solutions of the reduced BCS and, concerning the one pair transfer matrix element, we display their divergence in the proximity of G_{cr} .

Finally, we prove that the existence of G_{cr} relates to the trace of \hat{H}_{pair} .

¹ Note, however, that, whereas the seniority v counts the fermions coupled to $J \neq 0$, the number v_l refers to $J = 0$ pairs.

2 Classification of the states

As is well known, the eigenvalues and the eigenvectors of the pairing Hamiltonian

$$\hat{H}_{\text{pair}} = \sum_{\nu=1}^L e_{\nu} \sum_{m_{\nu}=-j_{\nu}}^{j_{\nu}} \hat{a}_{j_{\nu}m_{\nu}}^{\dagger} \hat{a}_{j_{\nu}m_{\nu}} - G \sum_{\mu,\nu=1}^L \hat{A}_{\mu}^{\dagger} \hat{A}_{\nu}, \quad (1)$$

$\hat{A}_{\mu} = \sum_{m_{\mu}=1/2}^{j_{\mu}} (-1)^{j_{\mu}-m_{\mu}} \hat{a}_{j_{\mu},-m_{\mu}} \hat{a}_{j_{\mu}m_{\mu}}$ being the $J = 0$ pair destruction operator, are found by solving the Richardson equations [7]. These, for $n = 2$, reduce to the following system of two equations in the unknown E_1 and E_2 (referred to as pair energies):

$$1 - G \sum_{\mu=1}^L \frac{\Omega_{\mu}}{2e_{\mu} - E_1} + \frac{2G}{E_2 - E_1} = 0, \quad (2a)$$

$$1 - G \sum_{\mu=1}^L \frac{\Omega_{\mu}}{2e_{\mu} - E_2} + \frac{2G}{E_1 - E_2} = 0 \quad (2b)$$

or, equivalently, by adding and subtracting the above,

$$2 - G \sum_{\mu=1}^L \frac{\Omega_{\mu}}{2e_{\mu} - E_1} - G \sum_{\mu=1}^L \frac{\Omega_{\mu}}{2e_{\mu} - E_2} = 0, \quad (3a)$$

$$- \sum_{\mu=1}^L \frac{\Omega_{\mu}}{2e_{\mu} - E_1} + \sum_{\mu=1}^L \frac{\Omega_{\mu}}{2e_{\mu} - E_2} + \frac{4}{E_2 - E_1} = 0. \quad (3b)$$

In eqs. (2), (3) the degeneracy of the single-particle energy (s.p.e.) e_{μ} is Ω_{μ} and G is the strength of the pairing force. Since the Richardson equations deal with pairs of fermions (nucleons) coupled to an angular momentum $J = 0$, their eigenvalues, given by $E = E_1 + E_2$, are those of the zero-seniority states ($v = 0$). Importantly, these eigenvalues display different degrees of collectivity: hence, they are conveniently classified in terms of the latter. For this purpose, as mentioned in the introduction, we introduce a number v_l , that counts in a given state the number of particles prevented to take part into the collectivity, not because they are blind to the pairing interaction (indeed they are coupled to $J = 0$), but because they remain trapped in between the s.p.l., even in the strong-coupling regime. Specifically, we shall ascribe the value $v_l = 0$ to the fully collective state, $v_l = 2$ to a state set up with a trapped pair energy while the other displays a collective behaviour and $v_l = 4$ to the state with two trapped pair energies.

This classification is equivalent to the one of ref. [5], where the n pair energies are split (for any n) into two classes: those remaining finite as $G \rightarrow \infty$ and the others.

We shall explore this pattern of states in both the weak- and the strong-coupling regimes, commencing with the former, which is the simpler to deal with.

3 The weak-coupling domain

In the weak-coupling limit of course no collective mode develops, hence v_l has no significance. Adopting a perturbative treatment we write the pair energies E_1, E_2 as

$$E_i = 2e_{\mu_i} + Gx_i \quad (4)$$

(Gx_i being a perturbation). Hence

$$\sum_{\mu} \frac{\Omega_{\mu}}{2e_{\mu} - E_i} = -\frac{\Omega_{\mu_i}}{Gx_i} + \sum_{\mu \neq \mu_i} \frac{\Omega_{\mu}}{2(e_{\mu} - e_{\mu_i}) - Gx_i} \quad (5)$$

and, expanding in G , system (2) becomes

$$1 + \frac{\Omega_{\mu_1}}{x_1} - G \sum_{\mu \neq \mu_1} \frac{\Omega_{\mu}}{2(e_{\mu} - e_{\mu_1})} + \frac{2G}{2(e_{\mu_2} - e_{\mu_1})} + O(G^2) = 0, \quad (6a)$$

$$1 + \frac{\Omega_{\mu_2}}{x_2} - G \sum_{\mu \neq \mu_2} \frac{\Omega_{\mu}}{2(e_{\mu} - e_{\mu_2})} + \frac{2G}{2(e_{\mu_1} - e_{\mu_2})} + O(G^2) = 0, \quad (6b)$$

where the indices μ_1 and μ_2 select one configuration out of the unperturbed ones. At the lowest order in G , if $\mu_1 \neq \mu_2$, one has $x_i = -\Omega_{\mu_i}$ and the pair energies $E_i = 2e_{\mu_i} - G\Omega_{\mu_i}$ are real. Thus the energy eigenvalue $E = E_1 + E_2$ becomes

$$E = 2(e_{\mu_1} + e_{\mu_2}) - G(\Omega_{\mu_1} + \Omega_{\mu_2}). \quad (7)$$

Instead, if $\mu_1 = \mu_2$, entailing $\Omega_{\mu_1} > 1$, we have to solve the system

$$1 + \frac{\Omega_{\mu_1}}{x_1} - G \sum_{\mu \neq \mu_1} \frac{\Omega_{\mu}}{2(e_{\mu} - e_{\mu_1})} + \frac{2}{x_2 - x_1} + O(G^2) = 0, \quad (8a)$$

$$1 + \frac{\Omega_{\mu_2}}{x_2} - G \sum_{\mu \neq \mu_2} \frac{\Omega_{\mu}}{2(e_{\mu} - e_{\mu_2})} + \frac{2}{x_1 - x_2} + O(G^2) = 0. \quad (8b)$$

A generic solution reads

$$x_{1,2} = -(\Omega_{\mu_1} - 1) \pm i\sqrt{\Omega_{\mu_1} - 1}, \quad (9)$$

showing that E_1 and E_2 are always complex conjugate. Hence the energy becomes

$$E = 4e_{\mu_1} - 2G(\Omega_{\mu_1} - 1), \quad (10)$$

which is of course real.

In comparing (7) and (10) with the lowest energy of one pair, which, in the weak-coupling regime, is [1]

$$E = 2e_{\mu_1} - G\Omega_{\mu_1}, \quad (11)$$

one sees that, while (7) corresponds to the sum of two contributions like (11) (the two pairs ignore each other), in (10) the Pauli blocking effect appears.

4 The strong-coupling domain

Here is where v_l has a significance and we deal with the states with $v_l = 0$ and 2 (the $v_l = 4$ states are of minor physical interest).

4.1 $v_l = 0$

In this section we study the eigenvalue of $v_l = 0$ in the strong-coupling limit. Clearly, such a state arises from an unperturbed configuration with the two pairs in the lowest s.p.l. (if $\Omega_1 > 1$) or in the two lowest s.p.l. We already know that in the degenerate case the collective eigenvalue is

$$E = 4\bar{e} - 2G(\Omega - 1), \quad (12)$$

\bar{e} being the energy and Ω the degeneracy of the level. The above yields the leading-order contribution to the energy in the non-degenerate case and represents a good estimate when the spreading of the s.p.e. levels is small with respect to $G\Omega$, as it is natural to expect.

To show this, following [1], we introduce the new variables

$$x_i = \frac{E_i - 2\bar{e}}{G\Omega}, \quad (13)$$

where now

$$\bar{e} = \frac{1}{\Omega} \sum_{\mu=1}^L \Omega_{\mu} e_{\mu}, \quad \text{and} \quad \Omega = \sum_{\mu=1}^L \Omega_{\mu}. \quad (14)$$

We further define the variance

$$\sigma = \sqrt{\frac{1}{\Omega} \sum_{\nu} \Omega_{\nu} (e_{\nu} - \bar{e})^2} \quad (15)$$

and the skewness

$$\gamma = \frac{1}{\sigma^3 \Omega} \sum_{\nu} \Omega_{\nu} (e_{\nu} - \bar{e})^3 \quad (16)$$

of the level distribution; moreover, we introduce the expansion parameter

$$\alpha = \frac{2\sigma}{G\Omega}. \quad (17)$$

Then system (3) becomes

$$2 + \frac{1}{x_1} \left(1 + \frac{\alpha^2}{x_1^2} + \frac{\alpha^3}{x_1^3} \gamma + \dots \right) + \frac{1}{x_2} \left(1 + \frac{\alpha^2}{x_2^2} + \frac{\alpha^3}{x_2^3} \gamma + \dots \right) = 0 \quad (18a)$$

$$\frac{1}{x_1} \left(1 + \frac{\alpha^2}{x_1^2} + \frac{\alpha^3}{x_1^3} \gamma + \dots \right) - \frac{1}{x_2} \left(1 + \frac{\alpha^2}{x_2^2} + \frac{\alpha^3}{x_2^3} \gamma + \dots \right) + \frac{4}{\Omega} \frac{1}{x_2 - x_1} = 0. \quad (18b)$$

At the leading order system (18) is easily solved, yielding

$$x_{1,2}^{(0)} = -\frac{(\Omega - 1)}{\Omega} \pm i \frac{\sqrt{\Omega - 1}}{\Omega}. \quad (19)$$

Higher-order terms can be obtained through a recursive linearisation procedure. One gets

$$x_{1,2} = -\frac{(\Omega-1)}{\Omega} - \alpha^2 \frac{(\Omega-2)}{(\Omega-1)} + \alpha^3 \gamma \frac{(\Omega-4)}{(\Omega-1)} \\ \pm i \frac{\sqrt{\Omega-1}}{\Omega} \mp i \frac{1}{2} \alpha^2 \frac{\Omega}{(\Omega-1)^{3/2}} \\ \pm i \alpha^3 \gamma \frac{\Omega}{(\Omega-1)^{3/2}} + \mathcal{O}(\alpha^4). \quad (20)$$

Note that the two pair energies are always complex conjugate. So the system's total energy reads

$$\frac{E-4\bar{e}}{G\Omega} = -2 \frac{(\Omega-1)}{\Omega} - 2\alpha^2 \frac{(\Omega-2)}{(\Omega-1)} \\ + 2\alpha^3 \gamma \frac{(\Omega-4)}{(\Omega-1)} + \mathcal{O}(\alpha^4). \quad (21)$$

Since the collective energy of one pair of nucleons living in L levels in the strong-coupling regime is [1]

$$\frac{E-2\bar{e}}{G\Omega} = -1 - \alpha^2 + \gamma\alpha^3 + \mathcal{O}(\alpha^4), \quad (22)$$

one sees that, when $\Omega \gg 1$, eq. (21) becomes just twice the value (22) and, moreover, the imaginary part of x_1 and x_2 goes to zero as $1/\sqrt{\Omega}$. Thus, in this limit, the Pauli interaction between the two pairs vanishes, as expected: the two pairs behave as two free quasi-bosons condensed in a level whose energy is given by (22).

In table 1 we compare the result (21) with the exact one assuming that the two pairs live in the first L levels of a harmonic-oscillator well with frequency ω_0 . The $v_l = 0$ collective state arises from an unperturbed configuration with a pair in the lowest and a pair in the next-to-the-lowest s.p.l. (in units of the oscillator frequency the energy of such a configuration is 8). We see that the difference between the two results never exceeds $\sim 15\%$, even for $\tilde{G} = G/\hbar\omega_0$ as low as 0.1. Note also that the energy of the $v_l = 0$ state scales with the size of the well (*i.e.* it does not depend upon ω_0).

4.2 $v_l = 2$

In the absence of the coupling term (Pauli principle) in the Richardson equations (2) the eigenvalues of the $v_l = 2$ states could be simply obtained by adding the collective energy E_1 carried by one pair and the trapped energy E_2 carried by the other pair.

This situation is recovered in the very strong-coupling limit, where all the s.p.e. become essentially equal to \bar{e} and both \bar{e} and E_2 are very small with respect to E_1 . Indeed the first equation of system (2) then becomes

$$\frac{1}{G} + \frac{\Omega}{E_1} - \frac{2}{E_1} = 0, \quad (23)$$

yielding

$$E_1 = -G(\Omega - 2), \quad (24)$$

Table 1. Energies, in units of $\hbar\omega_0$, of the state $v_l = 0$ for different values of $\tilde{G} = G/\hbar\omega_0$ at the order 0, 2 and 3 in the expansion parameter α compared with the exact ones. The $L = 4$ s.p.e. levels and the associated degeneracies are those of a 3-dimensional harmonic oscillator, ω_0 being the harmonic-oscillator constant.

\tilde{G}	α	$E^{(0)}$	$E^{(2)}$	$E^{(3)}$	E^{exact}
0.1	0.89	11.2	8.21	6.01	7.25
0.2	0.44	7.4	5.91	5.36	5.30
0.3	0.29	3.6	2.61	2.36	2.31
0.4	0.22	-0.2	-0.95	-1.08	-1.11
0.5	0.18	-4.	-4.60	-4.69	-4.70
0.6	0.15	-7.8	-8.30	-8.36	-8.37
0.7	0.13	-11.06	-12.03	-12.07	-12.08
0.8	0.11	-15.4	-15.77	-15.81	-15.81
0.9	0.10	-19.2	-19.53	-19.56	-19.56
1.0	0.09	-23.0	-23.30	-23.32	-23.32
1.5	0.06	-42.	-42.20	-42.21	-42.21

namely the energy of the state with two pairs and $v = 2$ in the one-level problem. This result, setting a correspondence between states with $v_l = 2$ and $v = 2$, connects *seniority* and “*like-seniority*” (or the physics of a “broken” and a “trapped” pair).

In the non-degenerate case, denoting with $E_1^{(1)}$ and $E_1^{(\nu)}$ the first and the ν -th eigenvalues ($\nu \neq 1$) of the one-pair equation

$$1 - G \sum_{\mu=1}^L \frac{\Omega_{\mu}}{2e_{\mu} - E_1} = 0, \quad (25)$$

the coupling (Pauli) term in (2) can be approximated for large G as follows:

$$\frac{2G}{E_2 - E_1} \simeq \frac{2G}{E_2^{(\nu)} - E_1^{(1)}} \sim \frac{2G}{2e_{\nu} - 2\bar{e} + G\Omega} \sim \frac{2}{\Omega}. \quad (26)$$

Hence the equations of system (2) decouple and can accordingly be recast as

$$\frac{1}{G_{\text{eff}}^{(1)}} - \sum_{\mu} \frac{\Omega_{\mu}}{2e_{\mu} - E_1} = 0, \quad (27a)$$

$$\frac{1}{G_{\text{eff}}^{(2)}} - \sum_{\mu} \frac{\Omega_{\mu}}{2e_{\mu} - E_2} = 0, \quad (27b)$$

where

$$\frac{1}{G_{\text{eff}}^{(1)}} \equiv \frac{1}{G} + \frac{2}{E_2^{(\nu)} - E_1^{(1)}} \simeq \frac{1 + \frac{2}{\Omega}}{G}, \quad (28a)$$

$$\frac{1}{G_{\text{eff}}^{(2)}} \equiv \frac{1}{G} - \frac{2}{E_2^{(\nu)} - E_1^{(1)}} \simeq \frac{1 - \frac{2}{\Omega}}{G}. \quad (28b)$$

We thus see that in the strong-coupling regime the Pauli principle just re-scales the coupling constant, differently, however, for the collective and the trapped states:

Table 2. Exact and approximate (eq. (27)) energies of the states with $v_l = 2$ for $\tilde{G} = 5$. The s.p.e. and degeneracies are those of a 3-dimensional harmonic oscillator. All energies are in units of $\hbar\omega_0$. The number of levels considered is $L = 3$.

E_1^{exact}	E_2^{exact}	E_1	E_2
-33.325	3.307	-35.711	3.309
-33.988	5.720	-35.711	5.721

indeed the pairing interaction is quenched for the former and enhanced for the latter by the Pauli blocking.

We compare in table 2 the exact results with the approximate ones (eq. (27)) in the h.o. case for $L = 3$, in the strong-coupling regime ($\tilde{G} = 5$). In this example, five states exist: of these only two have $v_l = 2$, namely those associated with the unperturbed configurations having either one pair in the lowest s.p.l. and one in the highest or both pairs in the second s.p.l. From the table we see that the estimated trapped energies E_2 almost coincide with the exact ones, while the results for the collective energy E_1 are satisfactory (the error being $\simeq 7\%$).

In concluding this section, we observe that our results in the strong-coupling domain agree with the findings of ref. [8], where the problem of the pairing Hamiltonian for small superconducting grains is studied. Indeed, although in [8] the single-particle levels are non-degenerate, the conclusions of this paper are not affected by this assumption. Hence the results of ref. [8] for two pairs can be recovered from ours by setting $\Omega_\nu = 1$ in the formulae of this section. Likewise, our results can be directly derived from ref. [8] by setting $n = 2$ and by grouping the single-particle levels into L degenerate multiplets.

5 The critical value of G

In this section we first search for the critical values of the coupling constant G and for their expression in terms of the moments of the s.p.l. distribution for the $n = 2$ system (sect. 5.1). Next, in sect. 5.2, we discuss the physics of G_{cr} . Finally, in sect. 5.3, we compare the Richardson solution with the BCS one.

5.1 The analytic expression of G_{cr}

In the weak-coupling regime, when a state evolves from an unperturbed one with the two pairs living in the same level, then the pair energies E_1 and E_2 are always *complex conjugate*. On the other hand, when the state evolves from an unperturbed one having the two pairs living in two different s.p.l., then E_1 and E_2 are *real*.

By contrast, in the strong-coupling regime the pair energies E_1 and E_2 of the $v_l = 0$ state are always complex conjugate. It is then clear that, if the degeneracy Ω_1 of the lowest s.p.l. is greater than one, then the pair energies E_1 and E_2 of the $v_l = 0$ state are complex conjugate in both the weak- and strong-coupling regime and their behaviour with G is smooth.

On the other hand, if $\Omega_1 = 1$, since in the weak-coupling limit the two pairs must live on different levels, E_1 and E_2 are necessarily real in a neighbourhood of the origin, but become complex in the strong-coupling regime. Thus, a singularity in their behaviour as a function of G is bound to occur.

In this second case, it appears natural to surmise that the singularity takes place when E_1 and E_2 coincide. In fact in this case the Pauli term of the Richardson equations diverges, and it must be compensated by a divergence in the sum entering into system (2): this can only happen if E_1 or E_2 coincides with an unperturbed eigenvalue.

To find out the analytic expression of the critical value of G we start from a generic solution with two pair energies E_1 and E_2 , which evolve with G till they coincide at an unperturbed energy $2e_\nu$ of a s.p.l. with degeneracy Ω_ν . These pair energies must fulfill the G -independent equation (3b), that we cast in the form

$$(E_2 - E_1)^2 \sum_{\mu=1}^L \frac{\Omega_\mu}{(2e_\mu - E_1)(2e_\mu - E_2)} + 4 = 0 \quad (29)$$

and which allows to express E_2 as a function of E_1 . To render this relationship explicit we set

$$E_1 = 2e_\nu + x, \quad (30a)$$

$$E_2 = 2e_\nu + x\varphi_\nu(x), \quad (30b)$$

where $\varphi_\nu(x)$ is easily found to read, for vanishing x ,

$$\varphi_\nu(0) = \frac{\Omega_\nu - 2 \pm 2\sqrt{1 - \Omega_\nu}}{\Omega_\nu},$$

a real quantity only when $\Omega_\nu = 1$. This occurrence is crucial since, in order to compensate the divergence arising from the sum appearing in eq. (2a), namely $-G\Omega_\nu/(2e_\nu - E_1) = G\Omega_\nu/x$, the divergent Pauli term $2G/(E_2 - E_1) = 2G/[x(\varphi_\nu(x) - 1)]$ must be real.

Assuming then $\Omega_\nu = 1$, the link between the pair energies is easily found by inserting (30) into (29) and expanding in x . To fourth order in x , one obtains

$$E_2 = 2e_\nu - x \pm 2\mathcal{P}_{(2)\nu}x^2 - 4\mathcal{P}_{(2)\nu}^2x^3 + \left(\pm 9\mathcal{P}_{(2)\nu}^3 + 2\mathcal{P}_{(3)\nu}^3 \pm \frac{\mathcal{P}_{(4)\nu}^4}{\mathcal{P}_{(2)\nu}} \right) x^4 + \mathcal{O}(x^5), \quad (31)$$

where E_1 is hidden in x and the quantities

$$\mathcal{P}_{(k)\nu} = \left\{ \sum_{\mu=1(\mu \neq \nu)}^L \frac{\Omega_\mu}{(2e_\mu - 2e_\nu)^k} \right\}^{\frac{1}{k}} \quad (32)$$

are the inverse moments of the level distribution.

To compute G_{cr} we recast one of the equations of system (2) as follows:

$$1 + \frac{G}{x} - G \sum_{\mu=1(\mu \neq \nu)}^L \frac{\Omega_\mu}{2e_\mu - 2e_\nu} - \frac{G}{x} \mp \mathcal{P}_{(2)\nu}G = 0, \quad (33)$$

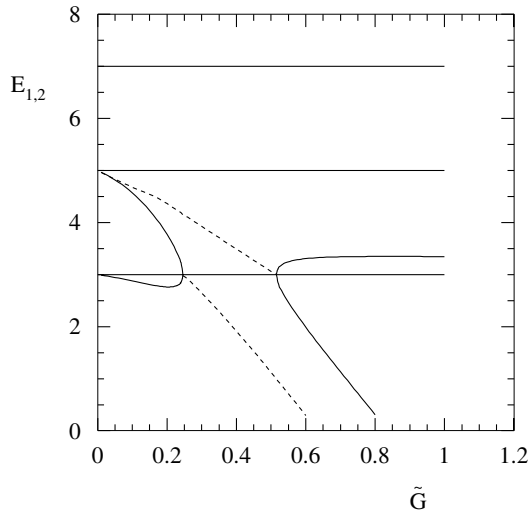


Fig. 1. Behaviour with \tilde{G} of the pair energies of the ground ($v_l = 0$) and first excited ($v_l = 2$) states obtained as solutions of the Richardson equations for a harmonic-oscillator well for $L = 3$: the dashed lines denote the real part of the solutions (of course coincident) when the solutions are complex and the solid lines the separate real parts. The pair energies are in units of $\hbar\omega_0$. As explained in the text, the two critical points refer to the two different states.

which is valid in the $x \rightarrow 0$ limit and transparently displays the cancellation of the divergences. Clearly the two solutions of (33) are

$$G_{\text{cr}}^{(\nu)\pm} = \frac{1}{\mathcal{P}_{(1)\nu} \pm \mathcal{P}_{(2)\nu}} = \left[\sum_{\mu=1(\mu \neq \nu)}^L \frac{\Omega_\mu}{2e_\mu - 2e_\nu} \pm \sqrt{\sum_{\mu=1(\mu \neq \nu)}^L \frac{\Omega_\mu}{(2e_\mu - 2e_\nu)^2}} \right]^{-1}, \quad (34)$$

which actually correspond to *two* critical values for G . We thus recover the results found long ago by Richardson [6] through a somewhat different route.

The situation is portrayed in fig. 1, where the behaviour of the pair energies with \tilde{G} is displayed for a harmonic-oscillator well assuming $L = 3$. Clearly, in this case the only s.p.l. with $\Omega = 1$ is the lowest one, hence the pair energies E_1 and E_2 , real in the weak-coupling limit, coincide at the critical point $\tilde{G}_{\text{cr}}^{(1)+}$ (their common value being $2e_1$) and then become complex conjugate; the energy of the associated state evolves in the $v_l = 0$ collective mode. By contrast, for the $v_l = 2$ state, arising from the configuration with two pairs living on the second level (which is allowed for the harmonic-oscillator well), the two pair energies E_1 and E_2 are complex conjugate in the weak-coupling limit, coalesce into the energy $2e_1$ at the critical point $\tilde{G}_{\text{cr}}^{(1)-}$ and then become real. One of the two solutions remains trapped above $2e_1$, while the other evolves into a collective state: the sum of the two yields the energy of the $v_l = 2$ state.

It is thus plain that for the occurrence of a critical value of G , a s.p.l. with $\Omega = 1$ must exist. Such s.p.l. is the

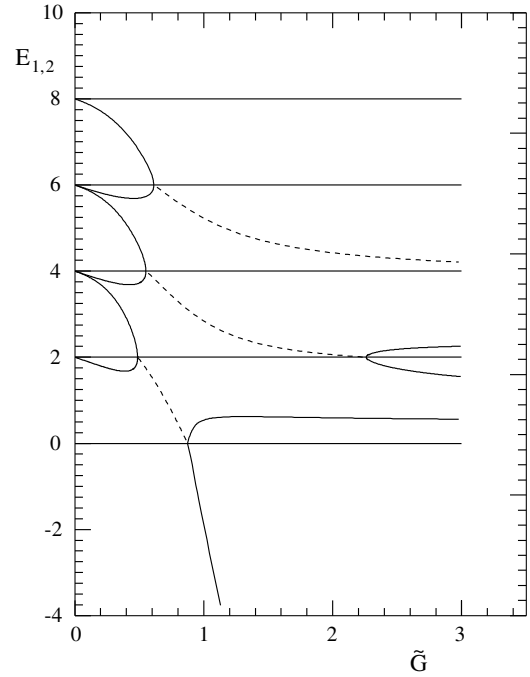


Fig. 2. Behaviour of the pair energies of three excited states in the case of $\Omega_\mu = 1$. The number of levels considered is $L = 9$.

lowest one in a harmonic oscillator, hence for this potential well both the ground and the first excited state of a $n = 2$ system carry one critical value of the coupling constant, namely $G_{\text{cr}}^{(1)+}$ and $G_{\text{cr}}^{(1)-}$, respectively. In correspondence of these G , the pair energies take on the value $E_1 = E_2 = 2e_1$. Thus for a $n = 2$ system two (at most) critical points exist on a $\Omega = 1$ s.p.l.

Finally, owing to the relevance of the $\Omega = 1$ degeneracy, we consider the model of L , for simplicity equally spaced, s.p.l. all having $\Omega = 1$, a situation occurring in metals and in deformed nuclei. In this instance two positive G_{cr} always exist in the lowest s.p.l. when $L \geq 3$ (in fact $G_{\text{cr}}^{(1)-} \rightarrow \infty$ for $L = 2$). Moreover, a positive $G_{\text{cr}}^{(1)-}$ implies complex E_1 and E_2 for $G < G_{\text{cr}}^{(1)-}$ and since for small G both the pair energies are real, they should evolve from an unperturbed configuration connected with the next higher-lying s.p.l., as illustrated in fig. 2. Numerically, we have found, for this model, that two G_{cr} appear on the second level when $L \geq 9$ and on the third level when $L \geq 16$. Thus in the $\Omega = 1$ model for two pairs to form, so to speak, a *quartet*, it is necessary that they live on adjacent s.p.l. in the unperturbed configuration. Furthermore, the more excited the configuration is the more not only G , but L as well, should be larger for the merging to occur, a fact clearly reflecting the competition between the mean field and the pairing force. Finally, observe that here, at variance with the finding of ref. [5, 9, 10], $G_{\text{cr}}^{(\nu+1)+} > G_{\text{cr}}^{(\nu)+}$: this is simply because we measure the s.p.e. from the bottom of the well rather than from the Fermi surface.

5.2 The significance of G_{cr}

In sect. 5.1 we have found that, when a s.p.l. with a degeneracy $\Omega = 1$ exists, at least two states of the $n = 2$ system carry a critical value G_{cr} . It is then natural to ask what is the relationship of the latter with the G_{cr} found in [1] where a system with $n = 1$ was addressed. Actually, in [1] G_{cr} marked the transition from a mean-field-dominated regime to the one ruled by the pairing interaction (as far as the trapped states are concerned). Moreover, in [1] we have proved the existence of G_{cr} for any potential well with s.p.l. of increasing degeneracy.

To connect the values of G_{cr} for the systems with $n = 1$ and $n = 2$, respectively, we consider again the harmonic-oscillator potential. For this well, in the $n = 1$ case, by explicitly computing the variance and the total degeneracy of the levels, the critical value of G , in units of $\hbar\omega_0$, is

$$G_{\text{cr}} \simeq \frac{8}{L^2} \quad (35)$$

when L is large. For the $n = 2$ system, first notice that, at large L , in $G_{\text{cr}}^{(1)\pm}$ the moment \mathcal{P}_1 dominates over \mathcal{P}_2 : hence, in this condition,

$$G_{\text{cr}}^{(1)+} \simeq G_{\text{cr}}^{(1)-} . \quad (36)$$

Moreover, again for L large enough,

$$\mathcal{P}_1 \simeq \frac{L^2}{8} \quad (37)$$

from where the equality

$$G_{\text{cr}}(n = 1) = G_{\text{cr}}(n = 2) \quad (38)$$

follows in the asymptotic L limit. From this outcome it follows that also when $n = 2$ the relevant dynamical element for $G < G_{\text{cr}}^{(1)+}$ is the mean field, whereas for $G > G_{\text{cr}}^{(1)-}$ is the pairing force, as far as the system's ground state is concerned.

One might ask how this interpretation can be reconciled with the findings related to the potential well of equally spaced s.p.l. with $\Omega = 1$, for which no G_{cr} occurs in the $n = 1$ system, whereas a $G_{\text{cr}}^{(+)}$ does exist in the $n = 2$ system.

To understand this occurrence, consider the structure of the trace of \hat{H}_{pair} . This, in fact, for the $n = 1$ system and *as far as the trapped modes are concerned* is found, in the strong-coupling limit, to read

$$\Sigma^{(\gamma)} = \frac{1}{L} \sum_{k=2}^L z_k = \frac{1}{\gamma + 2} , \quad (39)$$

where $z_k = (E_k - 2e_{k-1})/(2e_k - 2e_{k-1})$ actually measures the shift of each trapped pair energy E_k ($k = 2, \dots, L$) from the corresponding unperturbed energy $2e_k$. The above holds for any well with L (large) equally spaced s.p.l. with degeneracy

$$\Omega_k^{(\gamma)} \propto k^\gamma . \quad (40)$$

We thus see that (39) yields

$$\Sigma^{(\text{h.o.})} = \frac{1}{4} \quad (41)$$

for the harmonic oscillator ($\gamma = 2$) and

$$\Sigma^{(\Omega_k=1)} = \frac{1}{2} \quad (42)$$

for the $\Omega = 1$ model ($\gamma = 0$).

Hence for the latter the pairing force is so strongly active among the trapped pair energies to prevent the occurrence of a transition from the pairing to the mean-field-dominated physics, whereas the opposite happens for the harmonic-oscillator well.

In addition, the trace of \hat{H}_{pair} entails that in the $\Omega = 1$ model the collective mode is weaker than in the harmonic-oscillator case: indeed to set it up, not only a strong G , but a large degeneracy as well, is needed.

Concerning the $n = 2$ case in the $\Omega = 1$ model, the pairing dominance regime, defined by $G > G_{\text{cr}}^-$, is indeed postponed at values of G larger than in the harmonic-oscillator case. In fact one deduces from (34), in the large- L limit, the expression

$$G_{\text{cr}} \simeq \frac{2}{\log L} , \quad (43)$$

which, *en passant*, fixes the domain of validity of the perturbative expansion in G [11] for the $\Omega = 1$ model.

One might accordingly conjecture that no transition between different regimes will occur in this case among the trapped energies: we are currently performing the analysis of the behavior of the z_k *versus* G for the $v_l = 2$ states to ascertain whether this statement holds true.

5.3 Comparing the Bogoliubov quasi-particle and the Richardson exact solution

In this section we compare the exact solutions of the Richardson equations with the BCS solution in terms of Bogoliubov quasi-particles. To this scope, we self-consistently solve the well-known BCS equations for N fermions living in L levels, namely

$$v_\nu^2 = \frac{1}{2} \left[1 - \frac{\epsilon_\nu - \lambda}{\sqrt{(\epsilon_\nu - \lambda)^2 + \Delta^2}} \right] = 1 - u_\nu^2 , \quad (44)$$

$$\sum_{\nu=1}^L \frac{\Omega_\nu}{\sqrt{(\epsilon_\nu - \lambda)^2 + \Delta^2}} = \frac{2}{G} , \quad (45)$$

$$\sum_{\nu=1}^L \Omega_\nu \left[1 - \frac{\epsilon_\nu - \lambda}{\sqrt{(\epsilon_\nu - \lambda)^2 + \Delta^2}} \right] = N , \quad (46)$$

being $\Delta = G \sum_\nu \Omega_\nu u_\nu v_\nu$ the gap, λ the chemical potential and $\epsilon_\nu = e_\nu - G v_\nu^2$.

In the BCS framework the excitation energy of a system with seniority v is given by the energy of v quasi-particles, each carrying an energy

$$E_\nu^{\text{QP}} = \sqrt{(\epsilon_\nu - \lambda)^2 + \Delta^2} . \quad (47)$$

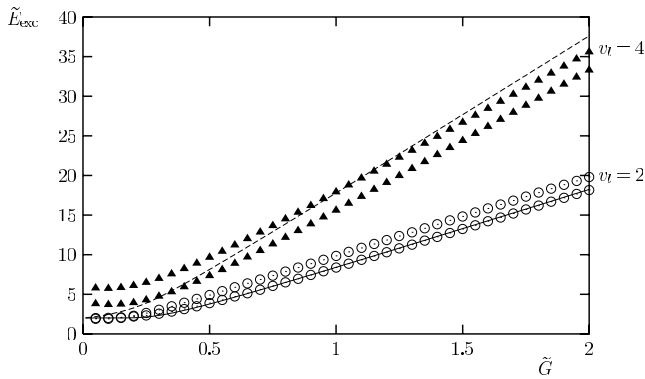


Fig. 3. Behaviour with \tilde{G} of the excitation energy of a two-pairs system in a harmonic-oscillator well with $L = 3$. The exact solutions with $v_l = 2$ (circles) and $v_l = 4$ (triangles) are compared with the BCS results for $v = 2$ (solid line) and $v = 4$ (dashed line).

In fig. 3 we display and compare the exact excitation energies $E_{\text{exc}} = E(v_l) - E(\text{g.s.})$ for a $L = 3$ harmonic-oscillator well and the corresponding Bogoliubov quasi-particles predictions for a $v = 2$ and a $v = 4$ state, whose excitation energies are $2E_1^{\text{QP}}$ and $2(E_1^{\text{QP}} + E_2^{\text{QP}})$, respectively. It appears that for \tilde{G} larger than the highest critical value ($\tilde{G}_{\text{cr}}^- \simeq 0.6$, see fig. 1) both the BCS and the exact excitation energies become linear functions of \tilde{G} and, remarkably, are very close to each other, in particular for $v_l = 2$. This result on the one hand shows that the BCS theory is a valid (at least for the lowest excitations) approximation of the exact physics even when the number of pairs and levels is indeed low, thus strengthening the correspondence between seniority and “like-seniority”. On the other hand, it confirms that the Richardson exact solutions behave like (47) in the strong- G limit, as found by Gaudin [4], who proved that for large L the excitation energies are given indeed by sums of terms like (47). It is remarkable that this appears to be approximately true already for $L = 3$.

Finally, we would like to remark that the exact energies displayed in fig. 3 correspond, microscopically, to two particles-two holes (the $v_l = 2$) and to four particles-four holes (the $v_l = 4$) excitations without the breaking of any pair. In other words, they are associated with the promotion of one or two pairs to higher-lying s.p.l.

It is also worth mentioning that while the s.p.e. ϵ_ν are almost constant in G , the quasi-particle energies E_ν^{QP} grow linearly with the latter (for $G > G_{\text{cr}}$) and their spreading in energy is lower than the one among the ϵ_ν .

We finally note that it would be interesting to compare the Richardson exact result with the energy of the excited 0_2^+ state known as “pairing vibrational” state calculated in the quasi-particle representation. This we leave for future work.

6 The wave functions

In this section we examine the wave functions of the states so far discussed.

As is well known, the $v = 0$ eigenfunctions of \hat{H}_{pair} for an n -pairs system are expressed in terms of the collective Grassmann variables Φ_μ (referred to as s -quasi-bosons), according to [12]

$$\psi_n[\Phi^*](m) = \prod_{k=1}^n \mathcal{B}_k^*(m), \quad (48)$$

where the set of indices (m_1, \dots, m_n) labeling the unperturbed configuration from where the state develops are collectively denoted by m . Since

$$\mathcal{B}_k^*(m) = \sum_{\mu=1}^L \beta_\mu^{(k)}(m) \Phi_\mu^* \quad (49)$$

can be viewed as the wave function of a pair, eq. (48) actually corresponds to the wave function of n independent pairs. The β coefficients are related to the eigenvalues E_k according to

$$\beta_\mu^{(k)}(m) = \frac{C_k(m)}{2e_\mu - E_k(m)}, \quad (50)$$

the $C_k(m)$ being normalisation factors. When no confusion arises, the index (m) will be dropped.

Since $(\Phi_\mu, \Phi_\nu^*) = \Omega_\mu \delta_{\mu\nu}$, it is convenient to replace the Φ 's and the β 's with

$$\tilde{\Phi}_\mu^* = \frac{\Phi_\mu^*}{\sqrt{\Omega_\mu}} \quad \text{and} \quad \tilde{\beta}_\mu^{(k)} = \sqrt{\Omega_\mu} \beta_\mu^{(k)}, \quad (51)$$

respectively. The normalisation of the state $|m\rangle$ then requires

$$C_k(m) = \frac{1}{\sqrt{\sum_\mu \frac{\Omega_\mu}{|2e_\mu - E_k(m)|^2}}}. \quad (52)$$

In the one-pair case, the above with $k = 1$ indeed entails $\langle(m)|m\rangle = 1$, whereas for a two-pairs state, (52) can still be used (of course, with $k = 1$ for the first- and $k = 2$ for the second-pair energy), but the associated state is no longer normalised to 1.

In [1] we have investigated the wave functions of a single pair living in many levels. We now study the behaviour with G of the wave functions of two pairs, our interest being focused on the states undergoing a transition, in the sense discussed in sect. 5.

For the sake of illustration, we take four s.p.l. of a harmonic-oscillator well and consider the states $v_l = 0$ (namely, the configuration $(m_1, m_2) = (1, 2)$) and one of the $v_l = 2$ (associated with the configuration $(m_1, m_2) = (2, 2)$). We display in fig. 4 and fig. 5, respectively, for different values of \tilde{G} , their coefficients $\tilde{\beta}_\mu^{(k)}$ as functions of the index μ .

In the $v_l = 0$ case, for $\tilde{G} < \tilde{G}_{\text{cr}}^{(1)+}$, the coefficients $\tilde{\beta}_\mu^{(k)}(1, 2)$ are real, while for $\tilde{G} > \tilde{G}_{\text{cr}}^{(1)+}$, they become complex conjugate (hence we display both their real and imaginary parts); the opposite occurs in the $v_l = 2$ case.

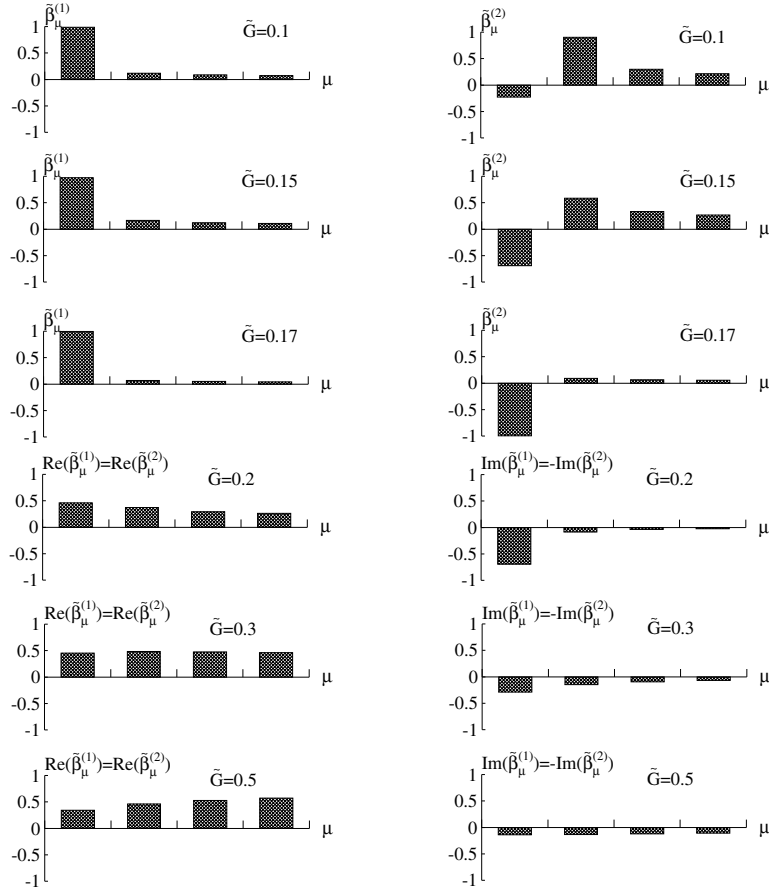


Fig. 4. In the $L = 4$ levels harmonic-oscillator well, we display for different values of \tilde{G} the coefficients $\tilde{\beta}_\mu^{(k)}$ (or their real and imaginary part) as functions of the index μ labeling the s.p.l. for the $v_l = 0$ state. The index k labels the pairs.

As appears in fig. 4, the $v_l = 0$ state initially has essentially one pair in the first and one in the second s.p.l. As \tilde{G} approaches $\tilde{G}_{\text{cr}}^{(1)+} = 0.171$, both pairs almost sit on the first level (thus displaying an apparent violation of the Pauli principle, which actually becomes a true violation at the critical point), where however the system cannot live. Finally, for $\tilde{G} \gg \tilde{G}_{\text{cr}}^{(1)+}$, the weight of all the components of $\tilde{\beta}^{(1)}(1, 2)$ and $\tilde{\beta}^{(2)}(1, 2)$ are almost the same, thus displaying a collective behaviour.

The $(2, 2)$ $v_l = 2$ state (see fig. 5) instead starts up with two pairs on the second s.p.l. (see the case $\tilde{G} = 0.1$ in fig. 5). As \tilde{G} approaches $\tilde{G}_{\text{cr}}^{(1)-} = 0.287$, the two pairs seem to live on the lowest single-particle level (again forbidden by the Pauli principle). Finally, for $\tilde{G} \gg \tilde{G}_{\text{cr}}^{(1)-}$, all the components of $\tilde{\beta}^{(1)}(2, 2)$ have almost the same weight (like the components of the collective state of one pair) while only the first component of $\tilde{\beta}^{(2)}(2, 2)$ is significant (as for the trapped state of one pair).

In conclusion, we provide analytic expressions for the $\tilde{\beta}_\mu^{(k)}$ coefficients for $G \rightarrow 0$, $G \rightarrow \infty$ and $G = G_{\text{cr}}$.

The weak-coupling limit is immediate and one obtains $\tilde{\beta}_\mu^{(1)}(m_1, m_2) = \delta_{m_1\mu}$ and $\tilde{\beta}_\mu^{(2)}(m_1, m_2) = \delta_{m_2\mu}$.

The strong-coupling limit is easily handled only when $v_l = 0$ (independently of Ω_1), when the s.p.e. are small with respect to E_1, E_2 . In fact one then gets

$$\tilde{\beta}_\mu^{(1,2)} = \frac{\sqrt{\Omega_\mu}}{\Omega} \left(\sqrt{\Omega - 1} \pm i \right). \quad (53)$$

The structure of the wave function at the critical points (assuming the transition to occur on the $\mu = 1$ level) is more delicate. To the order x^2 , one finds

$$\tilde{\beta}_1^{(1)} = - \left\{ 1 - \frac{x^2}{2} \mathcal{P}_{(2)1}^2 \right\}, \quad (54)$$

$$\tilde{\beta}_1^{(2)} = + \left\{ 1 - \frac{x^2}{2} \mathcal{P}_{(2)1}^2 \right\}, \quad (55)$$

$$\tilde{\beta}_{\mu \neq 1}^{(1)} = \sqrt{\Omega_\mu} \left\{ \frac{x}{2\epsilon_\mu - 2\epsilon_1} + \left(\frac{x}{2\epsilon_\mu - 2\epsilon_1} \right)^2 \right\} \quad (56)$$

$$\tilde{\beta}_{\mu \neq 1}^{(2)} = \sqrt{\Omega_\mu} \left\{ \frac{x}{2\epsilon_\mu - 2\epsilon_1} - \left(\frac{x}{2\epsilon_\mu - 2\epsilon_1} \right)^2 \right. \\ \left. \times [1 \pm 2\mathcal{P}_{(2)1}(2\epsilon_\mu - 2\epsilon_1)] \right\}. \quad (57)$$

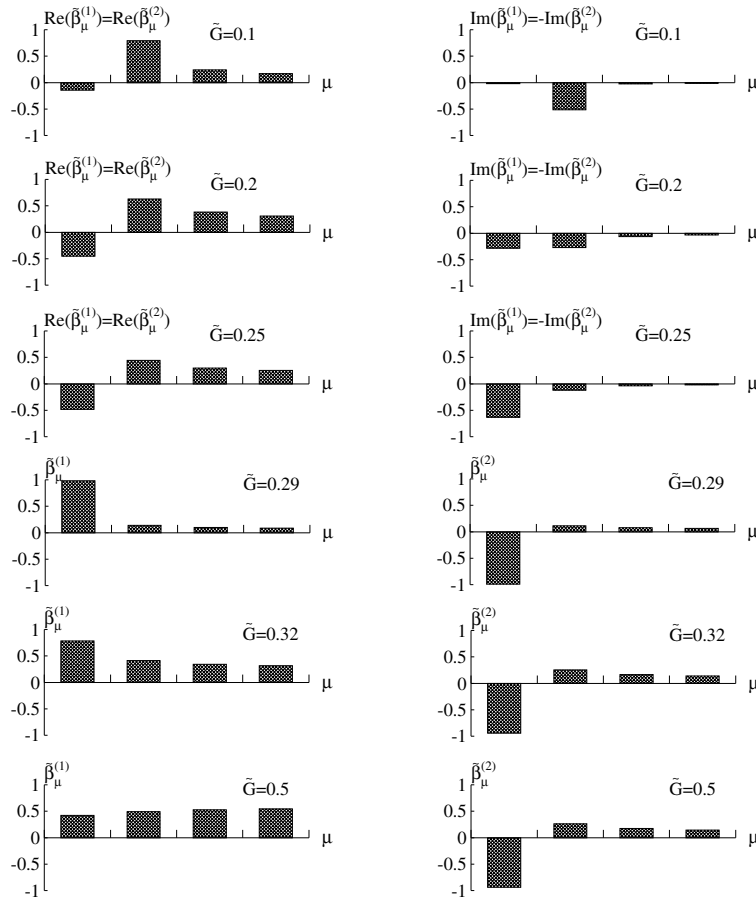


Fig. 5. The same as in fig. 4, but for the $v_l = 2$ state, made up of two pairs living on the second s.p.e. level.

In the above x is connected to G by the relation

$$x^2 = \mp 2(G - G_{\text{cr}}) \mathcal{P}_{(2)1} \frac{(\mathcal{P}_{(1)1} \pm \mathcal{P}_{(2)1})^2}{3\mathcal{P}_{(2)1}^4 + 4\mathcal{P}_{(2)1}\mathcal{P}_{(3)1}^3 + \mathcal{P}_{(4)1}^4} + \mathcal{O}\left((G - G_{\text{cr}})^2\right), \quad (58)$$

and in (57) the double sign refers to the two critical points $G_{\text{cr}}^{(1)\pm}$. Notice that the $\beta_{\mu \neq 1}^{(1,2)}$ start linearly in x , *i.e.*, in $\sqrt{G - G_{\text{cr}}}$, thus displaying a branch point in the control parameter G at its critical value.

7 The transition amplitudes

In this section we study the pair transfer amplitudes from the vacuum to a one-pair state and from a one-pair to a two-pairs state as a function of G , to explore how the transition between the two different regimes previously discussed is reflected in the matrix element.

We thus study the transition amplitude induced by the operator

$$\hat{A}^\dagger = \sum_{\mu} \hat{A}_{\mu}^\dagger \equiv \sum_{j_{\mu}} \sum_{m_{\mu}=\frac{1}{2}}^{j_{\mu}} (-1)^{j_{\mu}+m_{\mu}} \hat{a}_{j_{\mu}, m_{\mu}}^\dagger \hat{a}_{j_{\mu}, -m_{\mu}}^\dagger, \quad (59)$$

which enters into \hat{H}_{pair} , and examine the matrix elements

$$\langle 1 \text{ pair} | \hat{A}^\dagger | 0 \rangle \quad \text{and} \quad \langle 2 \text{ pairs} | \hat{A}^\dagger | 1 \text{ pair} \rangle. \quad (60)$$

In the fully degenerate case, the general expression for these transition amplitudes between states with any number n of pairs and seniority v , can be obtained by inserting into the pairing spectrum a complete set of states $|n', v'\rangle$. Indeed one has

$$\begin{aligned} \sum_{n', v'} \langle n+1, v | \hat{A}^\dagger | n', v' \rangle \langle n', v' | \hat{A} | n+1, v \rangle = \\ \left| \langle n+1, v | \hat{A}^\dagger | n, v \rangle \right|^2 = \\ \left(n+1 - \frac{v}{2} \right) \left(\Omega - n - 1 - \frac{v}{2} \right), \end{aligned} \quad (61)$$

thus getting for the transition amplitude

$$\langle n+1, v | \hat{A}^\dagger | n, v \rangle = \sqrt{\left(n+1 - \frac{v}{2} \right) \left(\Omega - n - \frac{v}{2} \right)}, \quad (62)$$

which, while more transparent, coincides with the one obtained in ref. [13]. Note the G -independence of (62), which, moreover, is diagonal in the seniority quantum number.

For large G , we could expect the matrix elements (60) to display an asymptotic behaviour coinciding with (62),

$$\langle(m_1, m_2)|\hat{A}^\dagger|(m)\rangle = \frac{\sum_{\mu\nu} [\tilde{\beta}_\mu^{(1)}(m_1, m_2)\tilde{\beta}_\nu^{(2)}(m_1, m_2)]^* \left[\tilde{\beta}_\mu(m)\sqrt{\Omega_\nu} + \tilde{\beta}_\nu(m)\sqrt{\Omega_\mu} - \frac{2}{\sqrt{\Omega_\mu}}\beta_\mu(m)\delta_{\mu\nu} \right]}{\mathcal{N}} \quad (69)$$

namely

$$\langle 1, v_l = 0 | \hat{A}^\dagger | 0 \rangle \xrightarrow{G \rightarrow \infty} \sqrt{\Omega}, \quad (63)$$

$$\langle 2, v_l | \hat{A}^\dagger | 1, v_l \rangle \xrightarrow{G \rightarrow \infty} \begin{cases} \sqrt{2(\Omega - 1)} & \text{for } v_l = 0, \\ \sqrt{\Omega - 2} & \text{for } v_l = 2, \end{cases} \quad (64)$$

if v_l is conserved (or if the above matrix elements are diagonal in v_l .)

However, the number v_l is not conserved at finite G : only in the strong-coupling regime the vanishing of the matrix element between states of different Gaudin numbers occurs.

Hence we now compute the transition amplitudes at finite G . For this purpose we first consider the transition from the vacuum to the one-pair state, namely

$$\langle(m)|\hat{A}^\dagger|0\rangle = \sum_{\mu} \sqrt{\Omega_{\mu}} \tilde{\beta}_{\mu}(m), \quad (65)$$

which, using (50) and the Richardson equations for one pair, namely

$$\sum_{\mu} \frac{\Omega_{\mu}}{2e_{\mu} - E} = \frac{1}{G}, \quad (66)$$

can be recast as follows:

$$\langle(m)|\hat{A}^\dagger|0\rangle = C_1(m) \sum_{\mu} \frac{\Omega_{\mu}}{2e_{\mu} - E(m)} = \frac{C_1(m)}{G}, \quad (67)$$

where the label (m) identifies the unperturbed configuration of the state.

The strong-coupling behaviour of (67) is different in the $v_l = 0$ and the $v_l = 2$ case. For the latter, the energy $E(m)$, for $G \rightarrow \infty$, remains trapped in between two s.p.l.: hence the G -dependent normalisation constant $C_1(m)$ will tend to a finite value $C_1^{\infty}(m)$. Accordingly, the matrix element (67) will vanish with G .

On the contrary, in the $v_l = 0$ case, where the unperturbed pair lives on the lowest level, $E(1) \sim -G\Omega$. Hence, for large G , since $C_1^{\infty}(1) \sim G\sqrt{\Omega}$, one finds

$$\langle(1)|\hat{A}^\dagger|0\rangle \sim \sqrt{\Omega}, \quad (68)$$

in accord with (63). Thus, at large G , for this matrix element, v_l and v coalesce.

Next, we study the transition from one to two pairs, namely the matrix element of \hat{A}^\dagger between $\mathcal{B}^*(m)$ and $\mathcal{B}_1^*(m_1, m_2)\mathcal{B}_2^*(m_1, m_2)$.

A lengthy calculation yields

see eq. (69) above,

the normalisation constant reading

$$\mathcal{N}^2 = 1 + \sum_{\mu\nu} \left[\left(\tilde{\beta}_{\mu}^{(1)}(m_1, m_2) \times \tilde{\beta}_{\nu}^{(2)}(m_1, m_2) \right)^* \tilde{\beta}_{\mu}^{(2)}(m_1, m_2) \tilde{\beta}_{\nu}^{(1)}(m_1, m_2) - \frac{2}{\Omega_{\mu}} \left| \tilde{\beta}_{\mu}^{(1)}(m_1, m_2) \right|^2 \left| \tilde{\beta}_{\mu}^{(2)}(m_1, m_2) \right|^2 \delta_{\mu\nu} \right]. \quad (70)$$

It will be computed in appendix A.

Using the Richardson equations to get rid of the sums, we recast (69) as follows

$$\langle(m_1, m_2)|\hat{A}^\dagger|(m)\rangle = \frac{1}{\mathcal{N}} \frac{2C_1(m_1, m_2)C_2(m_1, m_2)C_1(m)}{G[(E_1(m_1, m_2) - E(m))[(E_2(m_1, m_2) - E(m))]]}, \quad (71)$$

which is real since E_1 and E_2 are either real or complex conjugate.

We display in fig. 6 the amplitudes (71) for the transitions from a $n = 1$ to a $n = 2$ system (divided, for obvious convenience, by $\sqrt{\Omega}$). We consider the $n = 1$ system either in the ground collective state ($m = 1$) or in the first-excited trapped state ($m = 2$). On the other hand, the $n = 2$ system is either in the $v_l = 0$ (namely $(m_1, m_2) = (1, 2)$) or in the $v_l = 2$ ($(m_1, m_2) = (2, 2)$) state. The calculation is performed for a harmonic-oscillator well with $L = 4, 10, 20$ and 40 .

To understand the behaviour of the curves, first consider the weak-coupling limit, namely (from eq. (69))

$$\langle(m_1, m_2)|\hat{A}^\dagger|(m)\rangle \xrightarrow{G \rightarrow 0} \frac{\sqrt{\Omega_{m_1}}\delta_{m_2m} + \sqrt{\Omega_{m_2}}\delta_{m_1m} - \frac{2}{\sqrt{\Omega_{m_1}}}\delta_{m_1m}\delta_{m_2m}}{\sqrt{1 + \left(1 - \frac{2}{\Omega_{m_1}}\right)\delta_{m_1m_2}}}. \quad (72)$$

The above yields:

- $\sqrt{3}$ for the transition $m = 1 \rightarrow (m_1, m_2) = (1, 2)$,
- 0 for the transition $m = 1 \rightarrow (m_1, m_2) = (2, 2)$,
- 1 for the transition $m = 2 \rightarrow (m_1, m_2) = (1, 2)$,
- 2 for the transition $m = 2 \rightarrow (m_1, m_2) = (2, 2)$.

The behaviour around the critical points (which, of course, depend upon L) is quite delicate since both the numerator and the denominator in (69) are singular when E_1 and E_2 tend to $2e_1$. In fact the numerator in (69)

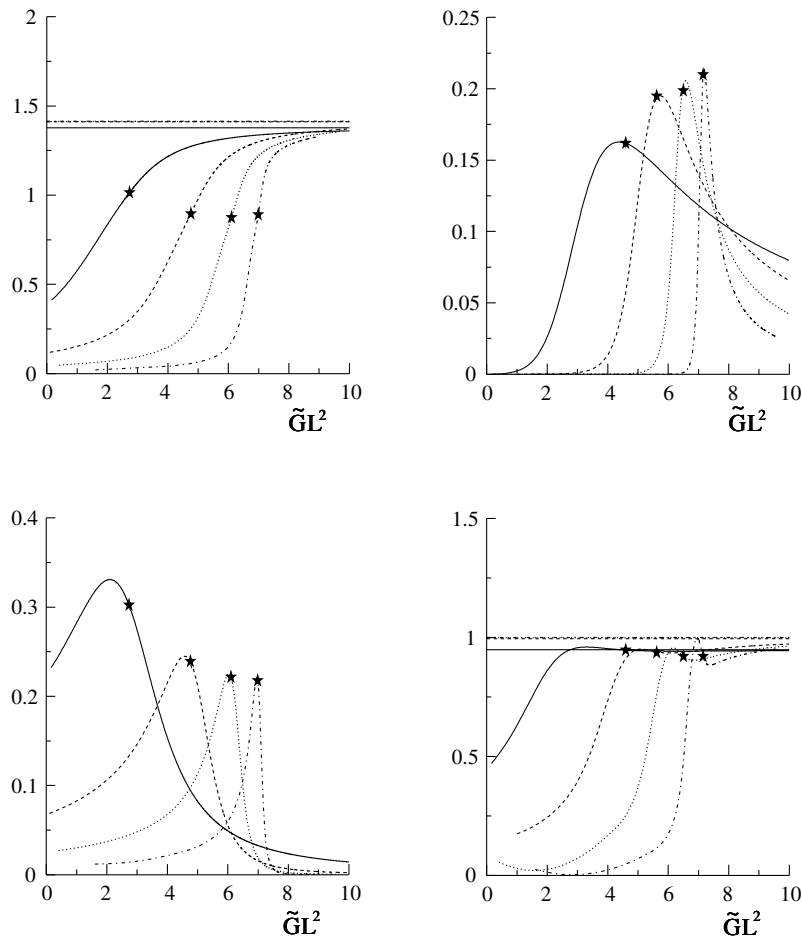


Fig. 6. The transition matrix elements (71) divided by $\sqrt{\Omega}$ for $L=4$ (solid), 10 (dashed), 20 (dotted) and 40 (dot-dashed) levels of a h.o. well. Upper left panel: transition from $m = 1$ (collective) state to $(m_1, m_2) = (1, 2)$ ($v_l = 0$) state; upper right panel: transition from $m = 1$ (collective) state to $(m_1, m_2) = (2, 2)$ ($v_l = 2$) state; lower left panel: transition from $m = 2$ (trapped) state to $(m_1, m_2) = (1, 2)$ ($v_l = 0$) state; lower right panel: transition from $m = 2$ (trapped) state to $(m_1, m_2) = (2, 2)$ ($v_l = 2$) state. A star denotes the position of the critical points. The straight lines represent the asymptotic values for $G \rightarrow \infty$.

vanishes like x^2 (see eqs. (30) for the definition of x), but so does the denominator. In the end the transition (69), for small x , is found to read

$$\langle (m_1, m_2) | \hat{A}^\dagger | (m) \rangle \xrightarrow{G \rightarrow G_{\text{cr}}^\pm} \frac{2 \sum_{\mu \neq 1} \frac{\sqrt{\Omega_\mu} \tilde{\beta}_\mu(m)}{2\epsilon_\mu - 2\epsilon_1} \pm \tilde{\beta}_1(m) \mathcal{P}_{(2)}}{G_{\text{cr}}^\pm \sqrt{6\mathcal{P}_{(2)}^4 \pm 8\mathcal{P}_{(2)}\mathcal{P}_{(3)}^3 + 2\mathcal{P}_{(4)}^4}}. \quad (73)$$

The value of the transition matrix elements at the critical value of G is marked by a star in fig. 6.

As we have seen in sect. 5.2, the inverse moments of the level distribution vanish in the $L \rightarrow \infty$ limit: hence (73) diverges. This occurrence, not appearing in fig. 6 because of the chosen normalisation, relates to the ODLRO (off-diagonal long-range order) which sets in into a system close to a phase transition.

Specifically, the diagonal amplitudes for large L behave according to

$$\langle (m_1, m_2) | \hat{A}^\dagger | (m) \rangle \stackrel{L \rightarrow \infty}{\simeq} \begin{cases} \sqrt{2\Omega} \theta(\tilde{G} - \tilde{G}_{\text{cr}}) & v_l = 0, \\ \sqrt{\Omega} \theta(\tilde{G} - \tilde{G}_{\text{cr}}) & v_l = 2. \end{cases} \quad (74)$$

The off-diagonal amplitudes behave instead as

$$\langle (m_1, m_2) | \hat{A}^\dagger | (m) \rangle \stackrel{L \rightarrow \infty}{\simeq} \delta(\tilde{G} - \tilde{G}_{\text{cr}}). \quad (75)$$

From the figures it is also clear that the critical value of G increases with L and $G_{\text{cr}}L^2 \rightarrow 8$ as $L \rightarrow \infty$ (see eq. (35)).

Finally, we consider the large- G limit, ruled by eqs. (71), (A.2) and (A.4).

Here first observe that the normalisation constant \mathcal{N} (see eq. (A.4) in the appendix) is always finite, reading

$$\mathcal{N} \rightarrow \begin{cases} \sqrt{2 - \frac{2}{\Omega}} & \text{for } v_l = 0, \\ \sqrt{1 - \frac{2}{\Omega}} & \text{for } v_l = 2. \end{cases} \quad (76)$$

In deducing the above, use has been made of the asymptotic expressions for E_1 and E_2 .

Considering next the transitions matrix element (71) from a one-pair to a two-pairs state with $v_l = 0$ or 2, we discuss the two cases:

i) Initial state: $n = 1$, $v_l = 0$.

For this, at large G , the energy behaves as $-G\Omega$ and $\mathcal{N} \sim G\sqrt{\Omega}$.

Now the $n = 2$ final state can have:

a) $v_l = 0$. In this case, using the expressions for the pair energies $E_{1,2}$ and the normalisation constants $C_{1,2}$, one finds the asymptotic result $\sqrt{2(\Omega - 1)}$, which coincides with the degenerate one for vanishing v (not v_l !);

b) $v_l = 2$. Here one solution (say E_1) behaves as $-G\Omega$, while the other (E_2) remains finite, as discussed in sect. 4.2. Specifically, one finds

$$E_1 \simeq -G_{\text{eff}}^{(1)}\Omega \simeq -\frac{\Omega^2 G}{\Omega + 2}$$

and E_2 much lower than E_1 and than the energy of the initial state. The normalisation constants are $C_1 = G_{\text{eff}}^{(1)}\sqrt{\Omega}$ and C_2^∞ (this one turns out to be G -independent and ~ 1). Using the above and the expression (76) for \mathcal{N} , one then gets for the matrix element the asymptotic value

$$\frac{C_2^\infty}{G} \sqrt{\frac{\Omega}{\Omega - 2}},$$

which is vanishing as G^{-1} in the strong-coupling limit, as it should, since it entails a variation $\Delta v_l = 2$ of ‘‘like-seniority’’.

ii) Initial state: $n = 1$, $v_l = 2$.

For this the asymptotic values of C and of the energy (denoted by C^∞ and E^∞) are finite: C^∞ is of order 1 and E^∞ is of the order of the unperturbed levels. Again, the $n = 2$ final state may have:

a) $v_l = 0$. In this case E^∞ is negligible with respect to E_1 and E_2 , whose asymptotic behaviour is known. The matrix element is then found to read

$$\frac{\sqrt{2}C^\infty}{G\sqrt{\Omega(\Omega - 1)}}.$$

Again, a transition $\Delta v_l = 2$ entails a behaviour of order G^{-1} , but in this case the extra factor $\sim \Omega^{-1}$ induces a faster decrease of the transition matrix element, as clearly apparent in fig. 6 (lower left panel);

b) $v_l = 2$. Calling E_2^∞ (and likewise C_2^∞) the $G \rightarrow \infty$ values of the pair energy (and of the normalisation constant) of the trapped pair, we find the following expressions for the asymptotic matrix element:

$$\frac{2C_2^\infty C^\infty}{\sqrt{\Omega - 2}G(E^\infty - E_2^\infty)}.$$

Now two different possibilities occur: if E^∞ and E_2^∞ are not both confined in the range $(2e_{\mu-1}, 2e_\mu)$

then, in the strong-coupling regime, their difference remains finite and the same occurs for the normalisation coefficients C^∞ and C_2^∞ . Hence the matrix element is ruled by the factor $1/G$ and thus vanishes. Otherwise, the difference $E^\infty - E_2^\infty$ tends to vanish. In fact, let $E(G)$ be the solution of the one-pair equation (66). Then the trapped pair energy E_2 , in the strong-coupling limit, solves eq. (27b), and, as a consequence, one has $E_2 = E(G_{\text{eff}}^{(2)})$, with $G_{\text{eff}}^{(2)}$ given by eq. (28). Thus, inserting $E_2 = E - y$ into eq. (27b) and expanding in y , one finds, using (66),

$$y = \frac{2|C^\infty|^2}{G\Omega}.$$

Hence the $G \rightarrow \infty$ limit of the transition matrix element becomes

$$\frac{C_2^\infty}{C^\infty} \frac{\Omega}{\sqrt{\Omega - 2}},$$

where the ratio C_2^∞/C^∞ can be shown to be given by

$$\frac{C_2^\infty}{C^\infty} = \frac{G}{G_{\text{eff}}^{(2)}} = 1 - \frac{2}{\Omega}.$$

Therefore the asymptotic matrix element reads $\sqrt{\Omega - 2}$, namely the value of the degenerate case (63).

8 Conclusions

It is well known that the pairing Hamiltonian, a reduced version of the BCS model of superconductivity, is solved by the Richardson equations as far as the states of zero seniority v are concerned.

In this paper, we have explored the solutions of these equations in the simple case of only two pairs of fermions, but living in any number of s.p.l. of any potential well. In this framework a further classification of the states in terms of v_l , which accounts for their degree of collectivity, appears convenient. It is elegant that v_l and v coincide for $G \rightarrow \infty$, whereas at finite G one (v_l) classifies the states according to the same pattern of the other (v) without breaking any pair. At small G , of course, no need for v_l is felt.

Concerning the structure of the solutions we have naturally searched, and obtained, an expression for the energy of the collective $v_l = v = 0$ mode in terms of the statistical features of the s.p.l. distribution, namely in terms of the moments of the latter. The same we have achieved for the two critical values of the coupling which mark the transition from the physics of the mean field to the one of the pairing force.

According to the general theory of Richardson, for a two-pairs system at most two critical values (G_{cr}^+ and G_{cr}^-) can exist on an unperturbed s.p.l. provided the pair degeneracy of the latter is one. Remarkably, G_{cr}^+ and G_{cr}^- are found to merge in the $L \rightarrow \infty$ limit and here coincide with

the G_{cr} , previously found in [1], for a system of just one pair living on any number of s.p.l. Note that in the latter case, a G_{cr} should *not* exist according to Richardson, but we have found that in fact it does by following the behaviour of the trapped modes with G . Its existence essentially rests on the trace of \hat{H}_{pair} or, equivalently, on the balance between the collective and the trapped modes, a balance reflecting not only the strength G of the pairing force, but, critically, the degeneracies of the s.p.l. In the $n = 2$ case, the latter markedly affect the critical values of G , which, in turn, signal the onset of the validity of the BCS framework. Indeed, for example, we have found that the G_{cr} of the $\Omega = 1$ model is much larger than the one of the harmonic-oscillator well. Whether this will entail the absence in the $\Omega = 1$ model of a transition in the trapped modes for the $n = 2$ system, as it happens for the $n = 1$ system, is an issue we are currently investigating.

Be as it may, we have explored in detail the transition associated to G_{cr}^+ and G_{cr}^- by following the behaviour with G of both the excitation energies of the $\nu_l = 2$ and $\nu_l = 4$ states and of the pair transfer matrix elements, finding for the latter a most conspicuous enhancement (the more so, the larger L) in the proximity of G_{cr} , clearly reflecting the onset of an ODLRO in the system. Concerning the excitation energies we have found that they closely approach the BCS predictions at large G especially for the case of two quasi-particle ($\nu_l = 2$) excitations, notwithstanding that the latter, in the BCS picture, amounts to break a pair, an occurrence never happening in the Richardson frame.

Finally, we have found that the BCS ground state of the $n = 2$ system in the simple $L = 3$ harmonic-oscillator model has $\Delta/d \simeq 2.5$ in the proximity of the critical values, being d the spacing among the s.p.l. (actually Δ/d is growing linearly with G). The Anderson criterion [14] for superconductivity is thus fulfilled. Indeed, in the ground state our system can be viewed as four fermions (a ‘‘quartet’’) sitting on the same energy level.

Appendix A.

In this appendix we shortly provide analytic expressions for the normalisation constants, whenever this is possible. When both the pair energies are real, no simple formulas for the normalisation coefficients $C_{1,2}(m, n)$ are available, whereas when $E_1^* = E_2$ one can use eq. (29) to get

$$C_{1,2} = \left| \frac{E_1 - E_2}{2} \right|. \quad (\text{A.1})$$

The above allows to simplify the expressions for the normalisation constant distinguishing, however, whether the solutions are real or complex conjugate.

In the former case, a tedious calculations provides

$$\mathcal{N} = \sqrt{1 - \frac{2[C_1^2(m, n) + C_2^2(m, n)]}{[E_2(m, n) - E_1(m, n)]^2}}, \quad (\text{A.2})$$

still in terms of the unspecified normalisation constants C_1 and C_2 .

In the latter case the normalisation is known, but, to get \mathcal{N} , we need to define

$$\tilde{C}_k(m, n) = \frac{1}{\sqrt{\sum_{\mu} \frac{\Omega_{\mu}}{(2e_{\mu} - E_k(m, n))^2}}}, \quad k = 1, 2 \quad (\text{A.3})$$

where, unlike in (52), no absolute values appear. Thus, the $\tilde{C}_k(m, n)$ are complex and $\tilde{C}_2^* = \tilde{C}_1$. With the help of (A.3), and using (A.1), one then gets

$$\mathcal{N} = \frac{|E_1 - E_2|^2}{4|\tilde{C}_1\tilde{C}_2|} \sqrt{1 - \frac{2(\tilde{C}_1^2 + \tilde{C}_2^2)}{(E_1 - E_2)^2}}. \quad (\text{A.4})$$

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